On the dimension of the space of integrals on coalgebras

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Abstract

We study the injective envelopes of the simple right C-comodules, and their duals, where C is a coalgebra. This is used to give a short proof and to extend a result of Iovanov on the dimension of the space of integrals on coalgebras. We show that if C is right co-Frobenius, then the dimension of the space of left M-integrals on C is $\leq \dim M$ for any left C-comodule M of finite support, and the dimension of the space of right N-integrals on C is $\geq \dim N$ for any right C-comodule N of finite support. If C is a coalgebra, it is discussed how far is the dual algebra C^* from being semiperfect. Some examples of integrals are computed for incidence coalgebras.

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0 Introduction and preliminaries

Integrals have played a key role in the structure and representation theory of Hopf algebras. The definition of integrals on Hopf algebras was given by Larson and Sweedler in [16]. Answering a question posed by Sweedler, Sullivan proved the uniqueness of integrals, i.e. that the dimension of the space of left (or right) integrals on a Hopf algebra is either 0 or 1. It turned out that the existence of integrals is closely related to some coalgebraic properties of the Hopf algebra. We recall that a coalgebra C is called right semiperfect if the category $\mathcal{M}^C$ of right C-comodules has enough projectives, or equivalently the injective envelope of any simple left C-comodule is finite dimensional. C is called right co-Frobenius if C embeds in C^* as a right C^*-module. We have similar notions to the left. The semiperfect property and the co-Frobenius property are not left-right symmetric. Also, if C is right (left) co-Frobenius, then C is right (left) semiperfect, while the converse is not true in general. However, for a

Hopf algebra H, we have that H is right semiperfect $\Leftrightarrow$ H is right co-Frobenius $\Leftrightarrow$ H is left semiperfect $\Leftrightarrow$ H is left co-Frobenius. Moreover, these are equivalent to H having non-zero left (or right) integrals (see [17]).

We recall that a left integral on a Hopf algebra H (over a field k) is an element T in the dual space H^* such that $h^*T = h^*(1)T$ for any $h^* \in H^*$. It is a simple, but very useful remark of Doi [9] that such a T is in fact a morphism of right H^*-modules from H to k. This suggests that one may consider integrals in a more general framework: if C is a coalgebra, then for any left C-comodule M, a left M-integral on C is a morphism of left C-comodules (or equivalently of right C^*-modules) from C to M, and for any right C-comodule N, a right N-integral on C is a morphism of right C-comodules (or equivalently of left C^*-modules) from C to N. We denote by $\int_{l,C,M}$ (respectively $\int_{r,C,N}$) the space of left M-integrals (respectively right N-integrals) on C. Using this point of view and a homological approach, Ştefan [20] proved that $\dim \int_{l,C,M} = \dim M$ for any finite dimensional left comodule over a right co-Frobenius coalgebra C which is either finite dimensional, or cosemisimple, or the underlying coalgebra structure of a Hopf algebra. In particular, Ştefan gave a short proof for the uniqueness of integrals in a Hopf algebra. Note that in all cases mentioned in the result above, C is left and right co-Frobenius. In [7] it is proved that for any right co-Frobenius coalgebra C which is also left semiperfect, we have that $\dim \int_{l,C,M} \leq \dim M$ for any finite dimensional left C-comodule M. In [15] Iovanov succeeded to prove this result in the case where C is only right co-Frobenius (thus to drop the left semiperfect condition), and moreover to prove that for such a C we also have that $\dim N \leq \dim \int_{r,C,N}$ for any finite dimensional right C-comodule N.

The main aim of this paper is to present a new approach to the dimension of the space of left or right integrals on a right co-Frobenius coalgebra, which produces a short proof of Iovanov's result, and in fact slightly extends this result. If C is a coalgebra, we say that a left C-comodule M, with comodule structure map $\rho : M \to C \otimes M$, has finite support if there exists a finite dimensional subspace X of C such that $\rho(M) \subseteq X \otimes M$ (this is equivalent to the fact that the coalgebra of coefficients of M is finite dimensional). We prove in Section 2 that if C is right co-Frobenius, then $\dim \int_{l,C,M} \leq \dim M$ (as cardinal numbers) for any left C-comodule M of finite support, and $\dim N \leq \dim \int_{r,C,N}$ for any right C-comodule N of finite support. Our approach to this result has as a main theme the study of the subspaces $\mathrm{Rat}({}_{C^*}C^*)$, $\mathrm{Rat}(C^*_{C^*})$, $\oplus_{j\in J} E(T_j)^*$ and $\oplus_{i\in I} E(S_i)^*$ of C^*, as well as the relations between these subspaces, where $C_0 = \oplus_{i\in I} S_i$ (respectively $C_0 = \oplus_{j\in J} T_j$) are decompositions of the coradical C_0 of C as a direct sum of left (respectively right) C-subcomodules, and by $E(M)$ we denote the injective envelope of a (left or right) C-comodule in the category of comodules. In the case where C is left and right semiperfect, these four spaces are equal. This study is done in Section 1. In the same section we explain how far is the dual C^* of a coalgebra from being a semiperfect ring. By using the rich topological structure of C^* (more precisely C^* is a complete topological ring with a certain topology determined by the coradical filtration of C), we show that idempotents lift modulo the Jacobson radical of C^* for any C. As a consequence, C^* is a semiperfect ring if and only the coradical of C is finite dimensional. As a

byproduct of our approach, we obtain in Section 2 that a right quasi-co-Frobenius coalgebra which is also a basic coalgebra is necessarily right co-Frobenius, and as an application we show that any right quasi-co-Frobenius coalgebra is Morita-Takeuchi equivalent to a right co-Frobenius coalgebra. In Section 3 we give some examples by computing the dimension of the spaces of left and right integrals on incidence coalgebras for certain comodules. It is known that incidence coalgebras are rarely right (or left) co-Frobenius, more precisely only when the underlying ordering relation is equality. We show that for a finite dimensional non-zero right comodule M over an incidence coalgebra C, the space of right M-integrals on C may be infinite dimensional, and also for any pair m, n of positive integers we can find examples such that M has dimension m, and the space of right M-integrals on C has dimension n.

We work over a fixed field k. If C is a coalgebra, then the comodule structure map of a left C-comodule M is denoted by $\rho : M \to C \otimes M$, $\rho(m) = \sum m_{-1} \otimes m_0$. Such a M is a right C^*-module with action of $c^* \in C^*$ on $m \in M$ denoted by $m \cdot c^*$. If R is a ring, and M, N are right (respectively left) R-modules, the set of morphisms of right (respectively left) R-modules from M to N is denoted by $\mathrm{Hom}_{-R}(M, N)$ (respectively $\mathrm{Hom}_{R-}(M, N)$). This notation will distinguish which side we are working in the case where M and N are R-bimodules. For basic definitions and facts about coalgebras we refer to [6], while for general facts about modules to [1].

1 Injective indecomposable comodules and their duals

Let C be a coalgebra. Then the coradical C_0 of C is the socle of the right C-comodule C, and also the socle of the left C-comodule C (see [6, Proposition 3.1.4]). We write $C_0 = \oplus_{j\in J} T_j$, as a direct sum of simple right C-comodules. Then for any $j \in J$ there exists an injective envelope $E(T_j)$ of T_j (in the category $\mathcal{M}^C$) such that $E(T_j)$ is a right subcomodule of C and $C = \oplus_{j\in J} E(T_j)$ (see [12] or [6, Theorem 2.4.16]). We have that $E(T_j)$ is an indecomposable injective object in $\mathcal{M}^C$. Then $C^* \simeq \prod_{j\in J} E(T_j)^*$ as right C^*-modules. In fact we identify C^* with $\prod_{j\in J} E(T_j)^*$, by regarding $E(T_j)^*$ as the set of all elements $c^* \in C^*$ such that $c^*(E(T_p)) = 0$ for any $p \in J - \{j\}$. Then we can also consider the right C^*-submodule $\oplus_{j\in J} E(T_j)^*$ of C^*, which is a dense subspace of C^* in the finite topology (see for example [6, Exercise 1.2.17]).

Let us note that the subspace $\oplus_{j\in J} E(T_j)^*$ of C^* does not depend on the representation $C_0 = \oplus_{j\in J} T_j$ as a direct sum of simple right comodules. Indeed, for each isomorphism type of simple right C-comodule there are only finitely many T_j's of that type, say $T_{j_1}, \ldots, T_{j_n}$, and $T_{j_1} \oplus \ldots \oplus T_{j_n}$ is just the (simple) subcoalgebra D of coefficients associated to that type of simple comodule. Then $E(T_{j_1})^* \oplus \ldots \oplus E(T_{j_n})^* = E(D)^*$, where $E(D)$ is the injective envelope of the right C-comodule D. Thus $\oplus_{j\in J} E(T_j)^* = \oplus_\lambda E(D_\lambda)^*$, where $C_0 = \oplus_\lambda D_\lambda$ is the decomposition of C_0 as a direct sum of simple subcoalgebras (and the D_λ's are uniquely determined by C_0).

For any $r \in J$ let $\varepsilon_r \in E(T_r)^* \subseteq C^*$ such that $\varepsilon_{r|E(T_r)} = \varepsilon$, and $\varepsilon_{r|E(T_j)} = 0$ for any $j \neq r$. It is easy to see that $\varepsilon_r c^* = c^*$ for any $c^* \in E(T_r)^*$, in particular $\varepsilon_r^2 = \varepsilon_r$, and also that $\varepsilon_r c^* = 0$ for any $c^* \in E(T_j)^*$, with $j \neq r$. As a consequence, we have that $E(T_r)^* = \varepsilon_r C^*$.

The following will be a key result in the sequel.

Proposition 1.1 *Let M be a left C-comodule. Then the map*

$$\gamma_M : M \to Hom_{-C^*}(\oplus_{j\in J} E(T_j)^*, M),\ \gamma_M(m)(c^*) = m \cdot c^*$$

is an injective linear map. Moreover, if M has finite support, then γ_M is a linear isomorphism.

Proof: Let $m \in M$ such $\gamma_M(m) = 0$. Write $\rho(m) = \sum m_{-1} \otimes m_0$, where ρ is the comodule structure map of M. Since $\oplus_{j\in J} E(T_j)^*$ is dense in C^*, there exists $c^* \in \oplus_{j\in J} E(T_j)^*$ which is equal to ε on all m_{-1}'s. Then $m = m \cdot \varepsilon = m \cdot c^* = \gamma_M(m)(c^*) = 0$, so γ_M is injective.

Assume now that M has finite support. Then there exists a finite subset J_0 of J such that $\rho(M) \subseteq (\oplus_{j\in J_0} E(T_j)) \otimes M$. Then for any $m \in M$ we have that $m \cdot (\sum_{j\in J_0} \varepsilon_j) = m$ and $m \cdot c^* = 0$ for any $c^* \in E(T_r)^*$ with $r \notin J_0$. Let $\phi \in Hom_{-C^*}(\oplus_{j\in J} E(T_j)^*, M)$, and let $m = \sum_{j\in J_0} \phi(\varepsilon_j)$. We show that $\phi = \gamma_M(m)$, which will prove that γ_M is also surjective.
If $c^* \in E(T_r)^*$ with $r \notin J_0$, then $\phi(\varepsilon_r) = \phi(\varepsilon_r^2) = \phi(\varepsilon_r) \cdot \varepsilon_r = 0$, hence

$$\begin{aligned}
\phi(c^*) &= \phi(\varepsilon_r c^*) \\
&= \phi(\varepsilon_r) c^* \\
&= 0 \\
&= m \cdot c^* \\
&= \gamma_M(m)(c^*)
\end{aligned}$$

If $c^* \in E(T_r)^*$ with $r \in J_0$, then

$$\begin{aligned}
\gamma_M(m)(c^*) &= m \cdot c^* \\
&= \sum_{j\in J_0} \phi(\varepsilon_j) \cdot c^* \\
&= \sum_{j\in J_0} \phi(\varepsilon_j c^*) \\
&= \phi(c^*)
\end{aligned}$$

the last equality holding since $\varepsilon_j c^* = 0$ for any $j \neq r$, and $\varepsilon_r c^* = c^*$.
We conclude that $\phi = \gamma_M(m)$, which ends the proof. ∎

Proposition 1.2 *With notation as above we have that* $\mathrm{Rat}({}_{C^*}C^*) \subseteq \oplus_{j\in J} E(T_j)^*$.

Proof: Let $c^* \in \mathrm{Rat}({}_{C^*}C^*)$. Then there exist finite families $(c_\alpha)_\alpha \subseteq C$ and $(c^*_\alpha)_\alpha \subseteq C^*$ such that $d^*c^* = \sum_\alpha d^*(c_\alpha)c^*_\alpha$ for any $d^* \in C^*$. Choose a finite subset F of J such that all c_α's lie in $\oplus_{j\in F}E(T_j)$.

Let $d^* \in C^*$ such that $d^*_{|E(T_r)} = \varepsilon$ for any $r \notin F$, and $d^*_{|E(T_r)} = 0$ for any $r \in F$. Then $d^*(c_\alpha) = 0$ for any α, so then $d^*c^* = \sum_\alpha d^*(c_\alpha)c^*_\alpha = 0$. On the other hand, if $r \notin F$ and $c \in E(T_r)$, then $\Delta(c) \in E(T_r)\otimes C$, so then $(d^*c^*)(c) = \sum d^*(c_1)c^*(c_2) = \sum \varepsilon(c_1)c^*(c_2) = c^*(c)$. Since $d^*c^* = 0$ we get that $c^*(c) = 0$, and this shows that $c^* \in \oplus_{r\in F}E(T_r)^* \subseteq \oplus_{j\in J}E(T_j)^*$. ∎

Let us note that $\mathrm{Rat}({}_{C^*}C^*)$ is a submodule of the left C^*-module C^*, while $\oplus_{j\in J}E(T_j)^*$ is a submodule of the right C^*-submodule C^*.

Similarly, if we work with left C-comodules, we can write $C_0 = \oplus_{i\in I}S_i$, a direct sum of simple left comodules, then $C = \oplus_{i\in I}E(S_i)$, a direct sum of indecomposable injective left C-comodules, and then we have that $\mathrm{Rat}(C^*_{C^*}) \subseteq \oplus_{i\in I}E(S_i)^*$, where $\mathrm{Rat}(C^*_{C^*})$ is a submodule of the right C^*-module C^*, while $\oplus_{i\in I}E(S_i)^*$ is a submodule of the left C^*-submodule C^*, which does not depend on the choice of the simples $(S_i)_i$ in the decomposition $C_0 = \oplus_i S_i$. The following result shows that under certain finiteness conditions on C, the subspaces $\mathrm{Rat}({}_{C^*}C^*)$, $\mathrm{Rat}(C^*_{C^*})$, $\oplus_{j\in J}E(T_j)^*$ and $\oplus_{i\in I}E(S_i)^*$ are in a special relation.

Proposition 1.3 *Let C be a right semiperfect coalgebra. Then*

$$\mathrm{Rat}(C^*_{C^*}) \subseteq \oplus_{i\in I}E(S_i)^* \subseteq \mathrm{Rat}({}_{C^*}C^*) \subseteq \oplus_{j\in J}E(T_j)^*$$

Proof: Since C is right semiperfect, each $E(S_i)$ is finite dimensional. Then $E(S_i)^*$ is a rational left C^*-module (see for example [6, Lemma 2.2.12]), so $E(S_i)^* \subseteq \mathrm{Rat}({}_{C^*}C^*)$. Now all follows from Proposition 1.2 and its left hand side version. ∎

Remark 1.4 *There is, of course, a version of Proposition 1.3 for left semiperfect coalgebras. These two results show that if C is left and right semiperfect, then*

$$\mathrm{Rat}({}_{C^*}C^*) = \mathrm{Rat}(C^*_{C^*}) = \oplus_{i\in I}E(S_i)^* = \oplus_{j\in J}E(T_j)^*$$

This fact is already known (see [2, Theorem 2.4]).

Part of the following result appears in [14, Lemma 1.4]. For completeness we include a proof, which is new and seems to be shorter than the one in [14].

Proposition 1.5 *Let C be a coalgebra, T a simple right C-comodule and $E(T)$ its injective envelope in the category $\mathcal{M}^C$. Then $E(T)^*$ is an indecomposable right C^*-module and $End_{-C^*}(E(T)^*)$ is a local ring. Moreover $E(T)^*$ is a projective cover of the simple right C^*-module T^*.*

Proof: It is enough to prove the statement for $T = T_j$, where $j \in J$. Let $j \in J$. The dual of the inclusion morphism $T_j \to E(T_j)$ is a surjective morphism of right C^*-modules whose kernel is $T_j^{\perp} = \{g \in E(T_j)^* |\ g(T_j) = 0\}$. Then we have an isomorphism of right C^*-modules $E(T_j)^*/T_j^{\perp} \simeq T_j^*$. Since clearly T_j^* is a simple right C^*-module, we obtain that $T_j^{\perp}$ is a maximal right submodule of $E(T_j)^*$. Therefore $\mathrm{Rad}(E(T_j)^*) \subseteq T_j^{\perp}$, where by $\mathrm{Rad}(M)$ we denote the Jacobson radical of a right C^*-module M.

It is known (see for example [6, Proposition 3.1.8]) that

$$\mathrm{Rad}(C^*) = C_0^{\perp} = \{c^* \in C^* |\ c^*(C_0) = 0\}$$

Since $C_0 = \oplus_{j \in J} T_j$ and C^* is identified with $\prod_{j \in J} E(T_j)^*$, we have that

$$\prod_{j \in J} T_j^{\perp} = C_0^{\perp} = \mathrm{Rad}(C^*) = \mathrm{Rad}(\prod_{j \in J} E(T_j)^*) \subseteq \prod_{j \in J} \mathrm{Rad}(E(T_j)^*) \subseteq \prod_{j \in J} T_j^{\perp}$$

which shows that both inclusions in the sequence of relations in the row above are equalities. In particular $\mathrm{Rad}(E(T_j)^*) = T_j^{\perp}$, thus $T_j^{\perp}$ is the unique maximal submodule of $E(T_j)^*$, and then clearly $T_j^{\perp}$ is superfluous in $E(T_j)^*$. Thus $E(T_j)^*$ together the dual of the inclusion morphism is a projective cover of T_j^*.

Now since $E(T_j)^*$ is projective (as a direct summand of C^*), we have by [1, Proposition 17.19] that $End_{-C^*}(E(T_j)^*)$ is a local ring, in particular $E(T_j)^*$ is indecomposable. ∎

Remark 1.6 *Let us note that the relation* $\mathrm{Rad}(E(T_j)^*) = T_j^{\perp}$*, that we showed in the proof of Proposition 1.5, is just a local (corepresentation) version of the fact that* $\mathrm{Rad}(C^*) = C_0^{\perp}$*.*

Concerning the existence of projective covers in the category $\mathcal{M}_{C^*}$, we can prove even more than what is obtained in Proposition 1.5. The following result shows that any right C^*-module which is the dual of a finite dimensional right C-comodule has a projective cover.

Proposition 1.7 *Let M be a finite dimensional right C-comodule. Then M^* has a projective cover in the category of right C^*-modules.*

Proof: Write the socle of M as $T_1 \oplus \ldots \oplus T_n$, a sum of simple right C-comodules. Then $E(M) = E(T_1) \oplus \ldots \oplus E(T_n)$, so we have an epimorphism of right C^*-modules $E(T_1)^* \oplus \ldots \oplus E(T_n)^* \simeq E(M)^* \to M^*$. Now proceed as in the proof of [1, Theorem 27.6]. Denote $\mathrm{Rad}(C^*) = \mathcal{J}$, the Jacobson radical of C^*. Then we have an epimorphism of right C^*-modules $E(T_1)^*/E(T_1)^*\mathcal{J} \oplus \ldots \oplus E(T_n)^*/E(T_n)^*\mathcal{J} \to M^*/M^*\mathcal{J}$, which shows that $M^*/M^*\mathcal{J}$ is a finite direct sum of C^*-modules of the form $E(T)^*/E(T)^*\mathcal{J}$ (and these are all simple since by [1, Proposition 17.19] we have that $\mathrm{Rad}(E(T)^*) = E(T)^*\mathcal{J} = T^{\perp}$). By Proposition 1.5, each $E(T)^*/E(T)^*\mathcal{J}$ has a projective cover, hence so does $M^*/M^*\mathcal{J}$. Let $f : P \to M^*/M^*\mathcal{J}$

be a projective cover. Then f induces a morphism $g : P \to M^*$ such that $f = g\pi$, where $\pi : M^* \to M^*/M^*\mathcal{J}$ is the natural projection. By Nakayama's Lemma, π is a superfluous epimorphism, and now using [1, Lemma 27.5], we obtain that $g : P \to M^*$ is a projective cover of M^*. ∎

We recall from [1] that a ring R is called semiperfect if $R/\mathrm{Rad}(R)$ is semisimple and idempotents lift modulo $\mathrm{Rad}(R)$ (or equivalently any simple right R-module has a projective cover). We are interested to see when is the dual algebra C^* of a coalgebra C a semiperfect ring. The following shows that the second condition in the definition of semiperfectness is always satisfied. In order to prove this, we use the rich topological structure of C^*.

Proposition 1.8 *Let C be a coalgebra. Then idempotents lift modulo $\mathrm{Rad}(C^*)$, the Jacobson radical of the dual algebra C^*.*

Proof: Consider the algebra filtration $\ldots \subseteq F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \ldots$ of C^*, where $F_{-n} = C_{n-1}^{\perp}$ for any positive integer n, and $F_n = C^*$ for any non-negative integer n. Then C^* is a topological ring with the family $(C_n^{\perp})_{n\geq 0}$ as a fundamental system of neighborhoods of 0 (see for example [18, Chapter D, Section I]). The completion of C^* in this topology is

$$\begin{aligned}
\widehat{C^*} &= \varprojlim_{p\in\mathbf{Z}} C^*/F_p \\
&= \varprojlim_{n\geq 0} C^*/C_n^{\perp} \\
&= \varprojlim_{n\geq 0} C_n^* \\
&= \varprojlim_{n\geq 0} Hom_{\mathcal{M}^C}(C_n, C) \\
&= Hom_{\mathcal{M}^C}(\cup_{n\geq 0} C_n, C) \\
&= Hom_{\mathcal{M}^C}(C, C) \\
&\simeq C^*
\end{aligned}$$

Therefore C^* is a complete topological ring.

Clearly any element $x \in \mathrm{Rad}(C^*)$ is topologically nilpotent, i.e. the sequence $(x^n)_n$ converges to 0, since $x^n \in (C_0^{\perp})^n \subseteq F_{-n}$ for any $n > 0$. Now we can apply [18, Lemma VII.1, page 312] and obtain that idempotents lift modulo $\mathrm{Rad}(C^*)$. ∎

Now we are able to characterize coalgebras for which the dual algebra is a semiperfect ring.

Proposition 1.9 *Let C be a coalgebra. Then the following assertions are equivalent.*
(i) *The dual algebra C^* is a semiperfect ring.*

(ii) *The coradical C_0 of C is finite dimensional.*
(iii) *There exist finitely many isomorphism types of simple right (or left) C-comodules.*

Proof: Write $C_0 = \oplus_{\lambda \in \Lambda} D_\lambda$, a direct sum of simple subcoalgebras. Then $C_0^* \simeq \prod_{\lambda \in \Lambda} D_\lambda^*$, a direct product of simple algebras. Therefore C_0^* is a semisimple algebra if and only if Λ is finite. Then (i)$\Rightarrow$(ii) follows, since if C^* is semiperfect, we have that $C^*/\mathrm{Rad}(C^*) = C^*/C_0^{\perp} \simeq C_0^*$ is semisimple, so Λ must be finite, and then C_0 is finite dimensional. Also, (ii)$\Rightarrow$(i) holds, since $C^*/\mathrm{Rad}(C^*) \simeq C_0^*$ is semisimple, and by Proposition 1.8 idempotents lift modulo the Jacobson radical of C^*.

The equivalence of (ii) and (iii) is clear, since each isomorphism type of a simple right C-comodule is associated to a unique simple subcoalgebra of C (its coalgebra of coordinates).
∎

We have recalled that a ring is semiperfect if and only if any simple right module over that ring has a projective cover. If C is an arbitrary coalgebra, we can describe all simple right C^*-modules having a projective cover.

Proposition 1.10 *Let C be a coalgebra. Then a simple right C^*-module M has a projective cover if and only if M is a rational C^*-module (i.e. the module structure comes from a simple left C-comodule structure on M).*

Proof: Assume that M has a projective cover P as a right C^*-module. Then by [1, Proposition 17.19] we have that $P\mathrm{Rad}(C^*)$ is the unique maximal submodule of P, thus $\mathrm{Rad}(P) = P\mathrm{Rad}(C^*)$, P is cyclic and $M \simeq P/\mathrm{Rad}(P)$. Then $P \oplus Q \simeq C^*$ as right C^*-modules for some C^*-module Q. Factoring out the Jacobson radical, we obtain that $P/\mathrm{Rad}(P) \oplus Q/\mathrm{Rad}(Q) \simeq C^*/\mathrm{Rad}(C^*)$, thus M embeds in $C^*/\mathrm{Rad}(C^*)$ as a right C^*-module.

But $C^*/\mathrm{Rad}(C^*) \simeq C_0^*$ as right C^*-modules, where C_0^* is viewed as a right C^*-module via the algebra map $C^* \to C_0^*$ dual to the inclusion map $C_0 \to C$. Now write $C_0 = \oplus_{\lambda \in \Lambda} D_\lambda$, a direct sum of simple subcoalgebras. Then $C_0^* \simeq \prod_{\lambda \in \Lambda} D_\lambda^*$, which is even an isomorphism of right C^*-modules. It is easy to see that the socle of the right C^*-module $\prod_{\lambda \in \Lambda} D_\lambda^*$ is $\oplus_{\lambda \in \Lambda} D_\lambda^*$, which is a rational right C^*-module, since each D_λ^* is rational, as the dual of a finite dimensional comodule (see [6, Lemma 2.2.12]). Therefore M is rational, too.

For the converse, assume that M is a simple right C^*-module which is rational, so M is a simple left C-comodule. Then $S = M^*$ is a rational simple left C^*-module, and $M \simeq S^*$ as right C^*-modules. Now everything follows from the fact that S^* has the projective envelope $E(S)^*$ by Proposition 1.5. ∎

2 Finiteness conditions and integrals on coalgebras

In this section we investigate the connection between the injective indecomposable left comodules and the duals of the injective indecomposable right comodules, and apply it to the study of integrals on co-Frobenius coalgebras.

Proposition 2.1 *Assume that C is a right co-Frobenius coalgebra. Then there exists an injective map $\phi : I \to J$ such that $E(S_i) \simeq E(T_{\phi(i)})^*$ as right C^*-modules for any $i \in I$.*

Proof: Since C is right co-Frobenius, it embeds in C^* as a right C^*-module. In fact this is an embedding of C in $\mathrm{Rat}(C^*_{C^*})$. Since C must be right semiperfect, we have by Proposition 1.3 that $\mathrm{Rat}(C^*_{C^*})$ embeds in $\oplus_{j\in J}E(T_j)^*$ as a right C^*-module.

Let $S_{i_1}, \dots, S_{i_n}$ be the objects of a given isomorphism type among the S_i's (they are in finite number since their sum is the simple subcoalgebra of coefficients associated to that type). Then $E(S_{i_1}) \oplus \dots \oplus E(S_{i_n})$ embeds as a right C^*-module in $\oplus_{j\in J}E(T_j)^*$. Since all $E(S_i)$'s are finite dimensional, $E(S_{i_1}) \oplus \dots \oplus E(S_{i_n})$ embeds in fact in $\oplus_{j\in J_0}E(T_j)^*$ for a finite subset J_0 of J. But each $E(S_i)$ and each $E(T_j)^*$ is a right C^*-module with a local endomorphism ring, so using Azumaya's Theorem we obtain that there exist $j_1, \dots, j_n \in J_0$ such that $E(S_{i_1}) \simeq E(T_{j_1})^*, \dots, E(S_{i_n}) \simeq E(T_{j_n})^*$. We define $\phi(i_1) = j_1, \dots, \phi(i_n) = j_n$. Proceeding the same for all isomorphism types of simple left C-comodules, we get a map $\phi : I \to J$ such that $E(S_i) \simeq E(T_{\phi(i)})^*$ for any $i \in I$. It is clear that ϕ is injective, since for non-isomorphic S_α and S_β we can not have $\phi(\alpha) = \phi(\beta)$. Indeed, otherwise we would have $E(S_\alpha) \simeq E(T_{\phi(\alpha)})^* \simeq E(T_{\phi(\beta)})^* \simeq E(S_\beta)$, and then S_α and S_β must be isomorphic as socles of their injective envelopes. ∎

Corollary 2.2 *Let C be a right co-Frobenius coalgebra. Then*
(1) C is a direct summand of $\oplus_{j\in J}E(T_j)^*$ as a right C^*-module.
(2) $\oplus_{i\in I}E(S_i)^*$ is a direct summand of C as a left C^*-module.

Proof: (1) Let $\phi : I \to J$ be an injective map whose existence was proved in Proposition 2.1. We have that $C = \oplus_{i\in I}E(S_i) \simeq \oplus_{i\in I}E(T_{\phi(i)})^*$, hence

$$\oplus_{j\in J}E(T_j)^* \simeq C \oplus (\oplus_{j\in J-\phi(I)}E(T_j)^*)$$

(2) Clearly $E(T_{\phi(i)})$ is finite dimensional for any $i \in I$, so then $E(S_i)^* \simeq E(T_{\phi(i)})$ as left C^*-modules. Therefore

$$C \simeq \oplus_{j\in J}E(T_j) = (\oplus_{i\in I}E(T_{\phi(i)})) \oplus (\oplus_{j\in J-\phi(I)}E(T_j)) \simeq (\oplus_{i\in I}E(S_i)^*) \oplus (\oplus_{j\in J-\phi(I)}E(T_j))$$

∎

Another interesting consequence of Proposition 2.1 is the following. In the case where C is a Hopf algebra, the result is already known, see [13, Proposition 3.7], and also [5].

Corollary 2.3 *Let C be a right co-Frobenius coalgebra. Then for any simple left C-comodule S, the injective envelope $E(S)$ has a unique maximal subcomodule.*

Proof: Proposition 2.1 shows that $E(S) \simeq E(T)^*$ for a simple right C-comodule T. But $E(T)^*$ has a unique maximal submodule by the proof of Proposition 1.5. ∎

Now we can prove the main result of the paper.

Theorem 2.4 *Let C be a right co-Frobenius coalgebra. Then*
(1) For any left C-comodule M of finite support we have that $\dim \int_{l,C,M} \leq \dim M$.
(2) For any right C-comodule N of finite support we have that $\dim N \leq \dim \int_{r,C,N}$.

Proof: (1) We know from Corollary 2.2 that C is a direct summand of $\oplus_{j\in J} E(T_j)^*$ as a right C^*-module. Then there is a surjective morphism of vector spaces

$$Hom_{-C^*}(\oplus_{j\in J} E(T_j)^*, M) \longrightarrow Hom_{-C^*}(C, M) = \int_{l,C,M}$$

But $Hom_{-C^*}(\oplus_{j\in J} E(T_j)^*, M) \simeq M$ by Proposition 1.1, and the result is proved.
(2) Similarly we have a surjective morphism of vector spaces

$$\int_{r,C,N} = Hom_{C^*-}(C, N) \longrightarrow Hom_{C^*-}(\oplus_{i\in I} E(S_i)^*, N)$$

By a left hand side version of Proposition 1.1, we have that $Hom_{C^*-}(\oplus_{i\in I} E(S_i)^*, N) \simeq N$ as vector spaces, and the result follows. ∎

Corollary 2.5 *Let C be a left and right co-Frobenius coalgebra. Then for any left (respectively right) C-comodule M of finite support we have that* $\dim \int_{l,C,M} = \dim M$ *(respectively* $\dim \int_{r,C,M} = \dim M$*).*

We recall from [10] that a coalgebra C is called right quasi-co-Frobenius if C embeds as a right C^*-module in a free C^*-module. This is equivalent to C being projective as a left C-comodule (see [10, Theorem 1.3]). It is known that a right co-Frobenius coalgebra is right quasi-co-Frobenius, and a right quasi-co-Frobenius coalgebra is right semiperfect, while the converse assertions are not true in general. We also recall that a coalgebra C is called basic if the coradical of C is a direct sum of pairwise non-isomorphic simple left coideals (see [3] or [4]). As a byproduct of the proof of Proposition 2.1 we obtain the following.

Proposition 2.6 *Let C be a right quasi-co-Frobenius coalgebra which is also a basic coalgebra. Then C is right co-Frobenius.*

Proof: Since C is basic, we have that $C_0 = \oplus_{i\in I} S_i$, a direct sum of pairwise non-isomorphic simple left C-subcomodules of C. As C is right quasi-co-Frobenius, it embeds as a right C^*-module in a free C^*-module, and then so does each $E(S_i)$. But $E(S_i)$ is finite dimensional, so it embeds in fact in $(C^*)^n$ for some positive integer n. Hence $E(S_i)$ embeds in $\mathrm{Rat}(C^*_{C^*})^n$, which itself embeds in $\oplus_{j\in J}(E(T_j)^*)^n$. Now with the Azumaya type argument as in the proof of Proposition 2.1, we see that $E(S_i) \simeq E(T_j)^*$ for some j, and in this way we obtain an injective map ϕ as in Proposition 2.1. Then $C = \oplus_{i\in I} E(S_i) \simeq \oplus_{i\in I} E(T_{\phi(i)})^*$ embeds as a right C^*-module in C^*, so C is right co-Frobenius. ∎

Two coalgebras are called Morita-Takeuchi equivalent if their categories of right (or equivalently of left) comodules are equivalent, see [22]. It is a classical result that any quasi-Frobenius finite dimensional algebra is Morita equivalent to a Frobenius algebra. The following is a dual version (and a generalization) of this result, for coalgebras of arbitrary dimension.

Corollary 2.7 *Any right quasi-co-Frobenius coalgebra is Morita-Takeuchi equivalent to a right co-Frobenius coalgebra.*

Proof: Let C be a right quasi-co-Frobenius coalgebra, and let D its basic coalgebra (see for example [4]), which is Morita-Takeuchi equivalent to C. Let $F :^C \mathcal{M} \to^D \mathcal{M}$ be an equivalence functor between the categories of left comodules. Then D embeds as a left D-comodule in a direct sum $F(C)^{(I)}$ of copies of $F(C)$, and as D is injective, it is a direct summand in $F(C)^{(I)}$. But $F(C)$ is projective, and then so are $F(C)^{(I)}$ and D, showing that D is also right quasi-co-Frobenius. Since D is also basic, Proposition 2.6 shows that it is right co-Frobenius. ∎

Remark 2.8 *Proposition 2.6 and Corollary 2.7 were proved in [11] in the particular case of coalgebras that are left and right quasi-co-Frobenius.*

3 Some examples of integrals on incidence coalgebras

Let $(X, \leq)$ be a locally finite partially ordered set, i.e. the interval $[x, y] = \{z | \ x \leq z \leq y\}$ is finite for any $x \leq y$. Let C be the incidence coalgebra of X, which has a linear basis $\{e_{x,y} | x, y \in X, x \leq y\}$, and comultiplication Δ and counit ε defined by

$$\Delta(e_{x,y}) = \sum_{x\leq z\leq y} e_{x,z} \otimes e_{z,y}$$

$$\varepsilon(e_{x,y}) = \delta_{x,y}$$

for any $x, y \in X$ with $x \leq y$ (where $\delta_{x,y}$ means Kronecker's delta).
Denote $x^+ = \{y|\ x \leq y\}$ and $x^- = \{y|\ y \leq x\}$ for any $x \in X$.

The only simple right (or left) C-subcomodules of C are the one-dimensional spaces $S_x =< e_{x,x} >$, where $x \in X$. The injective cover of the right C-comodule S_x is $E_r(S_x) =< e_{x,y}|\ y \in x^+ >$, while the injective cover of the left C-comodule S_x is $E_l(S_x) =< e_{y,x}|\ y \in x^- >$. Thus C is right (respectively left) semiperfect if and only if x^- (respectively x^+) is finite for any $x \in X$ (see for example [19]). It is known that C is very rarely co-Frobenius, more precisely it is right (or equivalently left) co-Frobenius if and only if the order relation on X is the equality (see [8]).

Let $M \in \mathcal{M}^C$ with comodule structure map ρ. Then giving a morphism of right C-comodules $f : E_r(S_x) \to M$ is the same with giving a family $(m_y)_{y \in x^+} \in M$ such that

$$\rho(m_y) = \sum_{z \in [x,y]} m_z \otimes e_{z,y} \text{ for any } y \in x^+ \tag{1}$$

Indeed, this correspondence associates to f the family $(f(e_{x,y}))_{y \in x^+}$.

Fix now some $u \in X$ and let $M = E_r(S_u)$. We will compute the right M-integrals on C. Let f be such an integral. Then for a fixed $x \in X$, the restriction of f to $E_r(S_x)$ is given by a family $(m_y)_{y \in x^+} \in E_r(S_u)$ satisfying equation (1). Write $m_y = \sum_{v \in u^+} \alpha_{y,v} e_{u,v}$ for some scalars $\alpha_{y,v}$ (only finitely many non-zero, for each y). Then equation (1) writes

$$\sum_{v \in u^+} \sum_{p \in [u,v]} \alpha_{y,v} e_{u,p} \otimes e_{p,v} = \sum_{z \in [x,y]} \sum_{r \in u^+} \alpha_{z,r} e_{u,r} \otimes e_{z,y} \tag{2}$$

We see that common tensor monomials in the left and right hand sides of equation (2) occur only for $z = p = r$ and $v = y$. Hence if it is not true that $u \leq y$, such common terms can not occur, and we have $\alpha_{y,v} = 0$ for any $v \in u^+$, which means that $m_y = 0$.

Assume now that $u \leq y$. Then the only common tensor monomials in the left and right hand sides are of the form $e_{u,p} \otimes e_{p,y}$ with coefficients $\alpha_{y,y}$ in the left hand side, and $\alpha_{p,p}$ in the right hand side, where $p \in [u,y] \cap [x,y]$. We obtain that $\alpha_{y,y} = \alpha_{p,p}$ for any such p, and $\alpha_{y,v} = 0$ for any $v \in u^+$, $v \neq y$. Moreover, if there exists $p \in [u,y] - [x,y]$ (and this is clearly equivalent to the fact that $u \notin [x,y]$), the term $e_{u,p} \otimes e_{p,y}$ shows up in the left hand side, but not in the right hand side, and we must have $\alpha_{y,y} = 0$, and then $m_y = 0$. In the case where such a p does not exist, i.e. $u \in [x,y]$, we must have $\alpha_{y,y} = \alpha_{u,u}$, a scalar not depending on $y \in u^+$. Therefore

$$f(e_{x,y}) = \begin{cases} 0, & \text{if } u \notin [x,y] \\ \alpha e_{u,y}, & \text{if } u \in [x,y] \end{cases}$$

where α is a scalar. This shows that

$$Hom_{\mathcal{M}^C}(E_r(S_x), E_r(S_u)) = 0 \text{ if } x \notin u^-$$

and

$$\dim Hom_{\mathcal{M}^C}(E_r(S_x), E_r(S_u)) = 1 \text{ if } x \in u^-$$

Then

$$Hom_{\mathcal{M}^C}(C, E_r(S_u)) \simeq \prod_x Hom_{\mathcal{M}^C}(E_r(S_x), E_r(S_u)) = \prod_{x \in u^-} Hom_{\mathcal{M}^C}(E_r(S_x), E_r(S_u)) \simeq k^{u^-}$$

In conclusion, if $M = E_r(S_u)$ we have that

• If u^+ is finite and u^- is infinite, then M is a finite dimensional right C-comodule, and $\int_{r,C,M}$ is infinite dimensional.

• If both u^+ and u^- are finite dimensional, then M is a right C-comodule of dimension $|u^+|$, and dim $\int_{r,C,M} = |u^-|$.

These show that for a non-zero finite dimensional right C-comodule M, the space $\int_{r,C,M}$ may be infinite dimensional, and also the pair (dim M, dim $\int_{r,C,M}$) may be any pair of positive integers (for certain choices of X and u).

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